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Information flows on hypergraphs

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Abstract

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We consider a generalization of the well-known gossip problem for hypergraphs: In a given set V of points, let each point of a subset $X \subseteq V$ know a unit of information which is not known by any other point. The points of V communicate by a sequence of k-party conference calls during which the k points exchange all the information known to them, and none of which are redundant. This exchange of information stops when every point of a subset $Y \subseteq V$ knows all the initial units of information. We give both lower and upper bounds on the number of calls in such a sequence, and we present a structural description of such processes.

1. Introduction

We consider combinatorial investigations of information exchanges in communication networks. The modern study of this topic began with the so-called 'gossip' or 'telephone' problem published in 1971 by Boyd [2]. Since then, a lot of papers on many different models of information dissemination in different networks have appeared. We refer the reader to the excellent 1988 survey article by Hedetniemi, Hedetniemi and Liestman [6] for a discussion of models and results, and for a very extensive list of references.

First we introduce a general model. Our definitions and notation follow the basic and very general ideas introduced in [4]. Let $\mathbb{N}$ be the set of positive integers, and let $2^{\mathbb{N}}$ be the set of all finite subsets of $\mathbb{N}$. Let n and k be arbitrarily fixed positive integers with $n \geqslant k \geqslant 2$, $V := \{1, 2, \ldots, n\}$, and E a collection of k-element subsets of V. The pair $H = (V, E)$ is called a k-uniform hypergraph, the elements of V and E are called the points and the edges of H, respectively.

Definition 1.1. An *information flow* (IF) on H is a mapping $\varphi : E \to 2^{\mathbb{N}}$ such that for any pair of distinct edges $e, e' \in E_1$, $e \cap e' \neq \emptyset$ implies $\varphi(e) \cap \varphi(e') = \emptyset$.

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The elements of $\varphi(e)$ are the *numbers* of e, and for any $e \in E$ and $r \in \varphi(e)$, the pair $c = (e; r)$ is said to be a *call* of φ. The rth *round* S_r is the set of all calls of φ with fixed number r. In this situation we also write that *c takes place on e during round r*. Each $v \in e$ is a *participant* of c, which we denote by $v \in c$. Similarly, for two calls $c = (e; r)$ and $c' = (e'; r')$, let $c \cap c' := e \cap e'$. Finally, let C be the set of all calls of φ.

Each call $c = (e; r)$ represents a k-party conference call at a fixed time unit (the round r) during which each of the k participants of c (the k points belonging to e) exchanges all information it knows at that moment with each other participant of c. By sequences of such calls we can organize the dissemination of initial items of information among the points of V. To formalize this we consider paths in H: For any given $x, y \in V$, a sequence $e_1, e_2, \ldots, e_m$ of edges $e_i \in E$ is called an (x, y)-path iff $x \in e_1$, $y \in e_m$ and $e_i \cap e_{i+1} \neq \emptyset$ $(i = 1, \ldots, m-1)$. Moreover, this path is said to be *φ-monotonic* iff there are numbers $r_i \in \varphi(e_i)$ $(i = 1, \ldots, m)$ with $r_1 < r_2 < \cdots < r_m$. Clearly, the calls $(e_1; r_1)$, $(e_2; r_2)$, $\ldots$, $(e_m; r_m)$ enable x to pass all information known to it before round r_1 to y.

Definition 1.2. For a k-uniform hypergraph $H = (V, E)$, let X and Y be nonempty subsets of V. An IF φ on H is called *(X, Y)-complete* iff for any $x \in X$ and any $y \in Y$, there is a φ-monotonic (x, y)-path. If $X = Y = V$, then φ is called *complete*.

According to our model, an (X, Y)-complete IF on H is a sequence of k-party conference calls such that each point of Y learns all items of information which at the beginning were known only by the points of X.

An (X, Y)-complete IF φ may contain calls which are not really necessary for (X, Y)-completeness. Our notion of redundance is: A call $c = (e; r)$ is redundant in the IF φ if the IF $\varphi - c$ is still (X, Y)-complete. Here $\varphi - c$ is defined by

$$\forall e' \in E: (\varphi - c)(e') := \begin{cases} \varphi(e') & \text{if } e' \neq e, \\ \varphi(e') - \{r\} & \text{if } e' = e. \end{cases}$$

Definition 1.3. An (X, Y)-complete IF φ on H is called *minimally (X, Y)-complete* iff it does not contain redundant calls.

For a given IF φ on a k-uniform hypergraph H the following parameters are of interest: $L(H, \varphi) := \sum_{e \in E} |\varphi(e)|$, i.e. the number of calls or *length* of the IF, $T(H, \varphi) := |\bigcup_{e \in E} \varphi(e)|$, i.e. the number of rounds or *time* of the IF. In this paper we are interested in the smallest and the largest length a minimally (X, Y)-complete IF on a k-uniform hypergraph on an n-element set V can have. In order to find these lengths we may restrict our general model to the case where H is the complete k-uniform hypergraph $(V, \binom{V}{k})$, i.e. E is the set of all k-element subsets of V. For given positive integers n and k, and subsets $X, Y \subseteq V$, define:

$$l := \min_{\varphi} L(H, \varphi) \quad \text{and} \quad L := \max_{\varphi} L(H, \varphi),$$

where $H = (V, \binom{V}{k})$ and the min/max is defined over all minimally (X, Y)-complete IF φ on H.

In Section 2 we investigate a modified minimal ordering associated with any IF. This leads to a structural description of IFs in the language of posets, which may be of some independent interest. In Section 3 we prove for the max-problem, in the case $X = Y = V$:

$$L = 2(n-k)+1 \qquad \text{if } n < n^*,$$

$$\left\lfloor \frac{n}{k} \right\rfloor^2 + 2\left(n - k\left\lfloor \frac{n}{k} \right\rfloor\right) \leqslant L \leqslant \frac{n^2}{k} - \left(2 - \frac{2}{k}\right)n \qquad \text{if } n \geqslant n^*,$$

where $n^* = 2k - 1 + [(2k-1)^2 - k(2k-1)]^{\frac{1}{2}}$.

Section 4 contains an enumeration result for posets related to IFs. This is used in Section 5 to prove the lower bound for the min-problem:

$$l \geqslant \left\lceil \frac{|X| - k}{k-1} \right\rceil + \left\lceil \frac{|Y| - k}{k-1} \right\rceil.$$

We also show that if $|X \cap Y| \geqslant k^2$, then equality holds above.

Finally, in Section 6 we discuss some existence results for posets representing the essential part of minimum length (X, Y)-complete IFs.

2. The reduced minimal order

The definitions and results in this section apply to the general model introduced in Section 1. Therefore, let a k-uniform hypergraph H on $V = \{1, 2, \ldots, n\}$ and an IF φ on H be given.

Then $R_M := \{((e;r), (e';r')) ;\ e \cap e' \neq \emptyset \text{ and } r \leqslant r'\}$ is a relation on the set C of calls. The transitive closure of R_M is called the *minimal order* (MO) of the calls of φ on H. It is denoted by $P_M = (C, \preccurlyeq)$. This definition was first given by Bumby [3]. By its MO the IF is completely described, because the transmission of information between the participants of two calls c and c' only depends on whether they are comparable in P_M or not: If $c \preccurlyeq c'$, then before c' takes place at least one participant of c' knows all the information exchanged during c.

The next step is to translate (X, Y)-completeness to orders, in order to do this let $X, Y \subseteq V$ be non-empty sets. Consider the subset C_v of all calls in which a given point v participates. If $C_v \neq 0$, then obviously it forms a chain in P_M, i.e. P_M induces a total (linear) ordering on C_v. Hence we can find exactly one minimum and one maximum element in C_v, which we denote by $\min v$ and $\max v$, respectively.

Definition 2.1. A partial order $P = (C, \preccurlyeq)$ on the set of calls is called (X, Y)-*complete*, iff for any $x \in X$, $y \in Y$ the sets C_x and C_y are non-empty and $\min x \preccurlyeq \max y$. In the case $X = Y = V$ we call P complete.

The following lemma suggests this definition.

Lemma 2.1. *P_M is (X, Y)-complete iff φ is (X, Y)-complete.*

Proof. If φ is (X, Y)-complete, then for any $x \in X$ and $y \in Y$, there is by Definition 1.2, a φ-monotonic (x, y)-path, whose calls form a chain in P_M. Let this chain start with call c and end with call d. Then $x \in c$, $y \in d$ and $c \leqslant d$. Because $\min x \leqslant c$, $d \leqslant \max y$, transitivity implies $\min x \leqslant \max y$, i.e. P_M is (X, Y)-complete.

To prove the converse, we have to construct a φ-monotonic (x, y)-path if $\min x \leqslant \max y$ is known. In order to do this, we find a maximal chain $\min x = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_m = \max y$. The order symbol together with a dot denotes the corresponding covering relation, i.e. $c \lessdot d$ iff $c < d$ but there is no c' with $c < c' < d$. Because $c_0 \cap c_1 \neq \emptyset$ there is a greatest call $d_1 \in \{c_1, \ldots, c_m\}$ such that $c_0 < d_1$ and $c_0 \cap d_1 \neq \emptyset$. Iterating this process we get a sequence $d_0 := c_0, d_1, \ldots, d_l$ of calls with $d_i < d_{i+1}$, $d_i \cap d_{i+1} \neq \emptyset$ but $d_i \cap d_j = \emptyset$ for $i = 0, 1, \ldots, l-1$, $j = i+2, \ldots, l-1$. This sequence ends with $d_l = c_m$ and the corresponding edges form a φ-monotonic (x, y)-path, since $x \in \min x = c_0 = d_0$ and $y \in \max y = c_m = d_l$. □

In the rest of this section let φ be minimally (X, Y)-complete. As an immediate consequence of Lemma 2.1 we know that for any $c \in C$, the MO of $\varphi - c$ is not (X, Y)-complete. Unfortunately, it is not possible to construct the MO of $\varphi - c$ directly from that of P_M without knowing the original IF φ. However we can infer the non-minimality of an IF by using the following corollary for checking a sufficient, but not necessary, condition for a call to be redundant in the MO.

Corollary 2.1. *If there is a call $c \in C$ such that for any $x \in X$ and any $y \in Y$, a chain from $\min x$ to $\max y$ not containing c can be found in the covering relation $(C; \lessdot)$, then c is a redundant call in the IF.*

Fig. 1 shows an example of a 2-uniform hypergraph (simple graph) F on 10 points and a minimal complete IF ρ on F. In Fig. 2 the corresponding MO is shown by its Hasse diagram. We omit the brackets and denote the calls by their participants.

The minimality of φ, and similarly of P_M, means that no call of C can be omitted without destroying (X, Y)-completeness. Although it makes no sense in the IF φ, in the poset P_M we can ask for minimality with respect to the covering relation instead of minimality with respect to the underlying set C: Can any relation $c \lessdot d$ be omitted from the covering relation $(C; \lessdot)$ such that the transitive closure of the remaining part is still an (X, Y)-complete order in the sense of Definition 2.1?

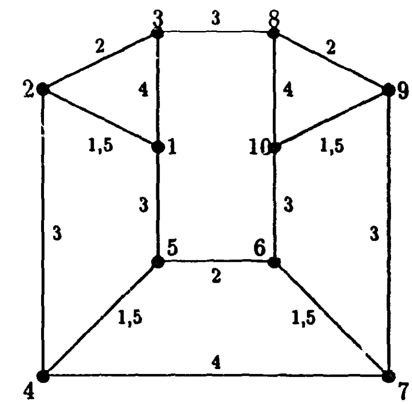

Fig. 1.

The example given in Fig. 1 and Fig. 2 already shows that this indeed is a new notion of minimality. Although the corresponding φ is minimally complete, after omitting $(\{4,5\};1) \lessdot (\{2,4\};3)$ we still find chains for any $\min x$ to any $\max y$ in the Hasse diagram, i.e. the transitive closure remains to be a complete order of the calls. So in the new sense, $(\{4,5\};1) \lessdot (\{2,4\};3)$ is redundant, and P_M is not minimally complete with respect to the covering relation. Because of this observation, we shall reduce the MO P_M to ensure the extended minimality, too.

To do this we need a third group of basic definitions which were suggested by an idea of Cot [5]. While our introductory definition is another than his, it will turn out the equivalence in the most important case later.

Definition 2.2. A call $c \in C$ is said to be an F^--call iff for any $y \in Y$, $c \preccurlyeq \max y$.

Let $\underline{F}^-$ denote the set of all F^--calls. For all $x \in X$, we have $\min x \in \underline{F}^-$ by the (X, Y)-completeness, i.e. in particular $\underline{F}^- \neq \emptyset$. Let F^- denote the set of all maximal elements of $\underline{F}^-$.

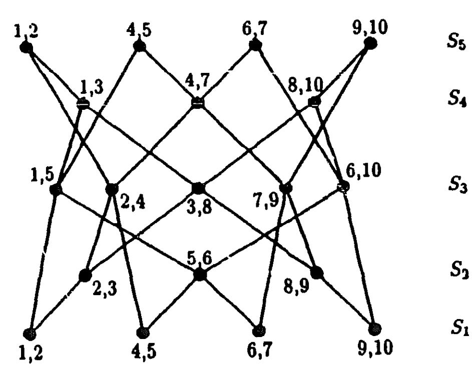

Fig. 2.

Definition 2.3. A call $c \in C$ is said to be an *F-call* iff for any $c' \in F^-$, $c' \leqslant c$.

Let $\bar{F}$ denote the set of all F-calls. Because of Definition 2.2, for each $y \in Y$, we have $\max y \in \bar{F}$. Hence $\bar{F} \neq \emptyset$ and we can denote the set of all minimal elements of $\bar{F}$ with F. The calls of F^- and F are called *irredundant* F^-- and F-calls, the remaining are called *redundant* F^-- and F-calls, respectively. The calls not included in the above sets form the *inner kernel* $K_0 := (\underline{F}^- \cup \bar{F})$. The *kernel* of the order is $K := K_0 \cup F^- \cup F$.

The following results are similar for $\underline{F}^-$ and $\bar{F}$ (in brackets). In what follows we use 'predecessor' and 'successor' in the sense of 'immediate' predecessor and successor, respectively.

Obviously, each call $a \in \underline{F}^- \setminus F^-$ $[a \in \bar{F} \setminus F]$ has at least one successor [predecessor] in $\underline{F}^-$ $[\bar{F}]$. We arbitrarily choose one of them and denote it by $s(a)$ $[p(a)]$. The finite chain

$$a \lessdot s(a) \lessdot s(s(a)) \lessdot \cdots \qquad [a \gtrdot p(a) \gtrdot p(p(a)) \gtrdot \cdots]$$

has its last element in F^- $[F]$. Denote this by $f^-(a)$ $[f(a)]$. For any given a, it is uniquely determined. Additionally, for $c \in F^-$ $[c \in F]$, we define $f^-(c) := c$ $[f(c) := c]$. Then altogether we have a unique mapping $f^- : \underline{F}^- \to F^-$ $[f : \bar{F} \to F]$. In particular, $a \leqslant f^-(a)$ $[f(a) \leqslant a]$ holds. For $x \in X$ $[y \in Y]$, we also use the abbreviation $f_x^- := f^-(\min x)$ $[f_y := f(\max y)]$.

Definition 2.4. The relation R_m on C is given by: For any $a, b \in C$, $(a, b) \in R_m$ iff
$a = b$
or $a, b \in \underline{F}^-$ and $b = s(a)$
or $a, b \in \bar{F}$ and $a = p(b)$
or $a, b \in K$ and $(a, b) \in R_M$.
The transitive closure $P_m := (C, \leqslant)$ of R_m is called the *reduced minimal order* (RMO) of the calls of φ on H.

Obviously the situation can occur that the RMO defined above is not uniquely determined for a given MO because it depends on the functions s and p. However any RMO will suffice in what follows. Except for Sections 4 and 6 which are devoted to general posets, we use the symbols $\leqslant$ and $\geqslant$ for an RMO.

Fig. 3 shows the RMO for F and ρ from Figs. 1 and 2. The irredundant F^-- and F-calls are drawn as □ and ■, respectively. For $k = 2$, note that the functions s and p are uniquely determined by the given MO, and so is the RMO.

We should remark that although $P_m \subseteq P_M$, on the kernel both are equal. This immediately implies that for any $c' \in F^-$ and $c \in F$, $c' \leqslant c$. Consequently, for any $x \in X$ and $y \in Y$, $f_x^- \leqslant f_y$ since $f_x^- \in F^-$ and $f_y \in F$. Because $\min x \leqslant f^-(\min x) = f_x^-$ and $\max y \geqslant f(\max y) = f_y$ we have $\min x \leqslant \max y$ for all $x \in X$ and $y \in Y$, i.e. P_m is (X, Y)-complete. This shows that indeed we have omitted redundant relations of the original MO.

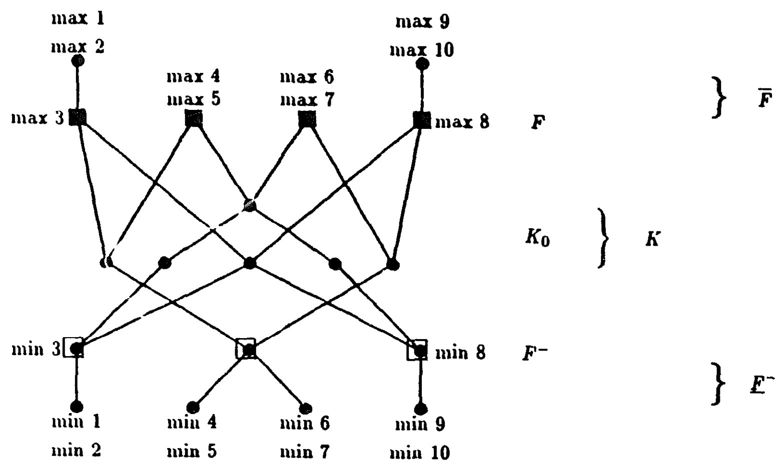

Fig. 3.

An easy consequence of the fact that $P_m \subseteq P_M$ is the validity of Corollary 2.1 even for the RMO, i.e. with $<$ instead of $\dot{<}$. We will use this fact to employ the minimality of φ.

In the remaining part of this section we list basic properties of the RMO which lead to a structural description of IFs and are used for the enumeration problems later on. Many of them are immedate consequences of the definitions or are well-known properties of posets. Thus technical details of the proofs are omitted. Because most of the properties hold similarly for $\underline{F}^-$ and $\bar{F}$, the statements for $\bar{F}$ are given in brackets.

Proposition 2.1. *F^- $[F]$ is an antichain, and $\underline{F}^-$ $[\bar{F}]$ is the lower [upper] ideal generated by F^- $[F]$.*

Proposition 2.2. *Let be $a \in \underline{F}^-$ $[a \in \bar{F}]$. If $c \in F^-$ and $a \leqslant c$ $[c \in F$ and $c \leqslant a]$, then $c = f^-(a)$ $[c = f(a)]$.*

Proposition 2.3. *For $a \in \underline{F}^-$ and $b \notin \underline{F}^-$ with $a \leqslant b$ $[a \in \bar{F}$ and $b \notin \bar{F}$ with $b \leqslant a]$, there is exactly one call $c \in F^-$ $[c \in F]$ with $a \leqslant c \leqslant b$ $[b \leqslant c \leqslant a]$, namely $f^-(a)$ $[f(a)]$.*

Proof. The uniqueness of the call $c \in F^-$ follows from the unique choice of a successor $s(a)$ for every $a \in \underline{F}^-$. To see that such a call exists, note that every maximal chain from a to b meets F^-. This contradicts the definition of R_m for otherwise one could find $a' \in \underline{F}^-$ and $b' \in \underline{F}^-$ with $a' \dot{<} b'$. □

For any $c \in F^-$ $[c \in F]$ let the *block* X_c $[Y_c]$ be the set of all points $x \in X$ with $f_x^- = c$ $[y \in Y$ with $f_y = c]$. Because of Propositions 2.1 and 2.2, $\bigcup_{c \in F^-} X_c$ $[\bigcup_{c \in F} Y_c]$ is a partition of X $[Y]$.

Proposition 2.4. *For $c \in F^-$ $[c \in F]$, the block X_c $[Y_c]$ is non-empty.*

Proof. Assume $X_c = \emptyset$. Then for $x \in X$ and $y \in Y$, the ascending chain from $\min x$ to $\max y$ meets the antichain F^- in $f_x^- \neq c$, i.e. none of these chains contains c. Hence c is redundant, which contradicts the minimality of φ. $\square$

We can use these results to give another characterization of $\underline{F}^-$ and $\bar{F}$.

Lemma 2.2. (a) $\underline{F}^- = \{a \in C: \forall y \in Y\ a \leq \max y\}$

(b) $\bar{F} \subseteq \{a \in C: \forall x \in X\ \min x \leq a\}$, *and if* $|F^-| \geq 2$, *then equality holds.*

Proof. (a) If $a \leq \max y$ for any $y \in Y$, then since $P_m \subseteq P_M$ we have $a \preccurlyeq \max y$, i.e. $a \in \underline{F}^-$ by definition. If $a \in \underline{F}^-$, then for any $y \in Y$, $a \leq f^-(a) \leq f_y \leq \max y$.

(b) Similarly, if $a \in \bar{F}$, then $\min x \leq f_x^- \leq f(a) \leq a$ for any $x \in X$. Now let $|F^-| \geq 2$ and $a \in C$ be given such that $\min x \leq a$ for any $x \in X$. If $a \in \underline{F}^-$, then $\min x \leq a \leq f^-(a)$, and X consists of the one block $X_{f^-(a)}$. But this contradicts our assumption that $|F^-| \geq 2$, i.e. we have $a \notin \underline{F}^-$. Because of Proposition 2.4, for $c \in F^-$, there is an element $x_0 \in X_c$. Since $\min x_0 \in \underline{F}^-$, $a \notin \underline{F}^-$, and $\min x_0 \leq a$, from Proposition 2.3 we know $\min x_0 \leq f_{x_0}^- = c \leq a$. Hence for all $c \in F^-$, $c \leq a$ and $a \in \bar{F}$. $\square$

The lack of symmetry between $\underline{F}^-$ and $\bar{F}$ turns out to be a consequence of the definition of these sets. For $n = 9$ and $k = 3$, Fig. 4 shows the RMO of a minimally complete IF, for which the call $(\{1, 4, 7\}; 2)$ is not an F-call, but it is greater than $\min x$ for each x. Hence, $|F^-| \geq 2$ is a necessary condition in Lemma 2.2(b).

If $|F^-| \geq 2$, then Lemma 2.2 shows the similarity between $\underline{F}^-$ and $\bar{F}$. Note that the inverse order P_m^- of P_m is minimally (Y, X)-complete and F^- and F of P_m

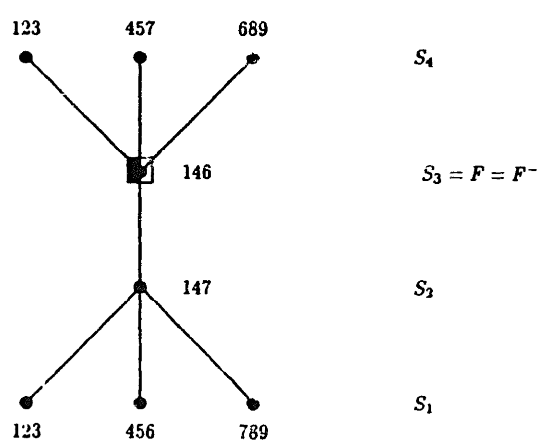

Fig. 4.

correspond to F and F^- of P_m^-, respectively. Therefore, throughout the paper, we may assume $|F^-| \leq |F|$ w.l.o.g. Moreover, if $|F^-| \geq 2$, Lemma 2.2 yields the following interpretation: F-calls are precisely those in which all items of information are available. The irredundant F-calls are the first of these F-calls. This is the definition of Cot [5]. F^--calls are the F-calls of the inverse IF.

The following theorem shows the simple structure of our RMO if $|F^-| = 1$.

Theorem 2.1. *The following statements are equivalent*:

(a) $\underline{F}^- \cap \bar{F} \neq \emptyset$,

(b) $F^- \cap F \neq \emptyset$,

(c) $|F^-| = 1$,

(d) $F^- = F$,

(e) $|F| = 1$.

Proof. (a)$\Rightarrow$(b); if $a \in \underline{F}^- \cap \bar{F}$, then $f^-(a) \geq a \geq f(a)$. Since $f^-(a) \in F^-$ and $f(a) \in F$, $f^-(a) \leq f(a)$, i.e. $f^-(a) = f(a) \in F^- \cap F$.

(b)$\Rightarrow$(c): If $c \in F^- \cap F$, then for any $x \in X$, $\min x \leq c \in F \subseteq \bar{F}$ by Lemma 2.2(b). Since $c \in F^-$, $X = X_c$, i.e. $F^- = \{c\}$.

(c)$\Rightarrow$(d), (e); If $F^- = \{c\}$, then by Definitions 2.3 and 2.4, $\bar{F} = \{d \in C : c \leq d\}$, i.e. $F = \{c\}$ generates this ideal.

(d)$\Rightarrow$(a): $\emptyset \neq F^- = F \subseteq \underline{F}^- \cap \bar{F}$.

(e)$\Rightarrow$(a): If $F = \{c\}$, then $Y = Y_c$. Therefore $c \leq \max y$ for all $y \in Y$, and from Lemma 2.2(a) we know $c \in \underline{F}^-$. Hence $c \in \underline{F}^- \cap F \subseteq \underline{F}^- \cap \bar{F}$. □

Theorem 2.1 suffices to handle the special case $|F^-| = 1$. We know that in this case only, the kernel of our IF collapses to a single call, i.e. the inner kernel is empty.

Let us, therefore, turn to the general case $|F^-| \geq 2$.

Proposition 2.5. *For any call $d \in K_0$, there are calls*

(i) *$c' \in F^-$ and $c \in F$ with $c' \leq d \leq c$;*

(ii) *$c' \in F^-$ and $c \in F$ with $c' \nleq d$ and $d \nleq c$.*

Proof. (i) Since d is not redundant, we can find a chain from $\min x$ to $\max y$ containing d. Then $f_x^- \leq d \leq f_y$ are the asserted calls.

(ii) If for any $c' \in F^-$, $c' \leq d$, then also for any $x \in X$, $\min x \leq d$. By Lemma 2.2(b) we conclude $d \in \bar{F}$ in contradiction to $K_0 = C \setminus (\underline{F}^- \cup \bar{F})$. □

As a consequence of the results above we additionally have that F^- and F are non-extendable antichains, each generating the RMO by its lower and upper ideal. To see this e.g. for F^-, note that for each call $a \in C$, one can find a call $c \in F^-$ with $a \leq c$ or $a \geq c$: if $a \in \underline{F}^-$, then $a \leq f^-(a) \in F^-$; if $a \in K_0$, then Proposition 2.5(i) applies; if $a \in \bar{F}$, then $a \geq f(a) \geq c$ for any $c \in F^-$.

Proposition 2.6. *Let $d \in K$ and $c \in F^-$. If there is some $x_0 \in X_c$ such that $\min x_0 \leqslant d$, then for any $x \in X_c$, $\min x \leqslant d$.*

Proof. We have $\min x_0 \leqslant f^-_{x_0} = c \leqslant d$, where Proposition 2.3 is used if $d \notin F^-$. But then for any $x \in X_c$, $\min x \leqslant c \leqslant d$. $\square$

We now close our investigation of the structural properties of P_m, summarizing our knowledge in a structural description of an IF. Note that according to the new concept of the RMO, the phrase 'y knows x's unit of information' now means that there is a call c such that $y \in c$ and $\min x \leqslant c$. Because $P_m \subseteq P_M$, this is different from the original meaning with respect to the minimal ordering $\leqslant$. A possible interpretation of the deletion of $(e; r) \lessdot (e'; r')$ from P_M is that the points of $e \cap e'$ do not save (store) all information handled in $(e; r)$ after round r. We refer to this with the phrase that the points of $e \cap e'$ 'forget' everything after round r. The interesting fact is that saving and further transmission of this set of items is not necessary for an (X, Y)-complete IF. Therefore, exchanging information according to the RMO means handling smaller sets of information units and saving memory space.

Structural description: Every minimally (X, Y)-complete IF consists of 3 phases any of which may be empty. The phases are completely separated by the irredundant F^-- and F-calls. During the first phase (the F^--calls) information is collected into blocks that are know to the participants in the irredundant F^--calls. All the other points can forget everything after this phase. The second phase (the kernel) consists of a transmission of information from the participants in irredundant F^--calls to those in irredundant F-calls, until each of the latter points knows everything. By Proposition 2.6 this is done block-wise, and moreover without any redundant F-calls. Again, after the kernel, all points not participating in any irredundant F-call may forget everything. They get the completely collected information in the redundant F-calls which form the third phase. Note that the original MO does not have these properties, in particular complete separation of the phases and block-wise transmission do not hold.

We should remark that the calls of the first or third phase are used only partially and in a 'directed' way: It is only necessary to transmit the information from all participants to one or from one to all the remaining, respectively. Therefore these calls are the generalized 'one-way-calls' introduced by Kleitman and Shearer [7]. The generalized 'two-way-calls' are the calls of the kernel. One also can consider the third phase as a collection of broadcasting processes on pairwise disjoint subsets, and the first phase an an analogous 'inverse' broadcasting. While the first and third phase are well-understood, the wide variety of distinct IFs and the difficulties arise from the kernel.

In order to give a good estimate for the number of calls in an IF we need one more property of K. For $a \in C$, let $\operatorname{suc} a$ and $\operatorname{pre} a$ denote the sets of successors and predecessors in the RMO, respectively.

Proposition 2.7. *Let* $|F^-| \geq 2$.
If $c \in F^- \cup K_0$, *then* $1 \leq |\text{suc}\, c| \leq k$.
If $c \in F \cup K_0$, *then* $1 \leq |\text{pre}\, c| \leq k$.

Proof. The lower bound follows from the definition of F^- and from Proposition 2.5(i) for $c \in K_0$. Assume $|\text{suc}\, c| > k$ for some $c \in F^- \cup K_0$. Note that $P_m = P_M$ on K. Because each successor has a non-empty intersection with c, we can find $c', c'' \in K$ such that $c \mathrel{\dot{<}} c'$, $c \mathrel{\dot{<}} c''$ and $c' \cap c'' \neq \emptyset$. But then c' and c'' must be in different rounds and consequently both are comparable, i.e. one of them does not cover c. This contradication proves $|\text{suc}\, a| \leq k$. The second inequality can be proved similarly. □

3. Long information flows

In this section we present the best-known upper bound for the number of calls in a minimally complete IF, i.e. for the case $X = Y = V$. Clearly, this is also an upper bound for a minimally (X, Y)-complete IF if X and Y are arbitrarily given.

At first, Burosch et al. [4] proved that for $k = 2$, $L \leq \binom{n}{2}$. In [12], Straßburg generalized the ideas and proved $L \leq \binom{n-k+2}{2}$ for arbitrary k.

Let a minimally complete IF φ on $(V, \binom{V}{k})$ be given arbitrarily. The basic idea of our approach is to estimate the number of calls in each of the three phases of the RMO constructed in Section 2.

Lemma 3.1. $|\underline{F}^- \setminus F^-| \leq n - k\,|F^-|$; $|\bar{F} \setminus F| \leq n - k\,|F|$.

Proof. Both inequalities can be proved similarly. Let us consider $\bar{F} \setminus F$. To avoid formalism let us consider the original IF and P_M. From Lemma 2.2 we know that in every call of $\bar{F}$ all items of information are transmitted. Therefore, after F, all information is known to the participants (points) of calls of F. Since F is an antichain in P_M, there are $k\,|F|$ such points. The remaining points get the complete set of information in the calls of $\bar{F} \setminus F$. Recall that a call $c = (e; r)$ is redundant if every participant in c knows everything before round r. Consequently, assigning to each of the $n - k\,|F|$ remaining points the smallest call of $\bar{F} \setminus F$ in which the point is a participant gives a surjective (unique) mapping. □

While Lemma 3.1 also holds for $|F^-| = |F| = 1$, we now suppose $|F^-| \geq 2$ for the following investigation of the inner kernel K_0.

Lemma 3.2. *For* $|F^-| \geq 2$, $|K_0| \leq (n - k)\,|F^-| - n$.

Proof. We use the RMO P_m and the interpretation given in the structural description at the end of Section 2. The main point is that in K_0 information is transmitted only block-wise because, by Proposition 2.6, after a given call $d \in K_0$, the initial items of each point of a block X_c ($c \in F^-$) are either known to all the participants of d or to none of them. Note that here 'known' again is used with respect to P_m as explained in Section 2! Along the same line, in the following we use the phrase "$v \in V$ *learns* the block X_c during the call $d \in K_0$" iff d is the (uniquely determined) minimal call which v participates in and which is greater than c. The asserted inequality is now proved by counting these situations twice. On the one hand, during every call $d \in K_0$, at least one point must learn at least one block, because otherwise this call would be redundant. On the other hand, after the calls of the inner kernel K_0, no point knows every block by Proposition 2.5(ii). The $k\,|F^-|$ participants of a call of F^- start K_0 with knowing one block already, and the remaining $n-k\,|F^-|$ points do not know any block before K_0. Hence,

$$|K_0| \leqslant k\,|F^-|\,(|F^-|-2) + (n-k\,|F^-|)(|F^-|-1) = (n-k)\,|F^-| - n. \qquad \square$$

Theorem 3.1. *For $X = Y = V$,*

$$L \leqslant \begin{cases} 2(n-k)+1 & \text{if } k \leqslant n < n^*, \\ \dfrac{n^2}{k} - \left(2 - \dfrac{2}{k}\right)n & \text{if } n \geqslant n^*, \end{cases}$$

where $n^* = 2k - 1 + [(2k-1)^2 - k(2k-1)]^{\frac{1}{2}}$.

Proof. For any minimally complete IF φ on H, if $|F^-| = 1$, then $|F| = |K| = 1$ by Theorem 2.1, and

$$|C| = |\underline{F}^- \setminus F^-| + |K| + |\bar{F} \setminus F| \leqslant 2(n-k)+1. \tag{1}$$

If $|F^-| \geqslant 2$, then

$$|C| = |\underline{F}^- \setminus F^-| + |F^-| + |K_0| + |F| + |\bar{F} \setminus F| \leqslant n + (n-2k+1)\,|F^-| - (k-1)\,|F|.$$

As a consequence of Lemma 2.2 we assumed $|F^-| \leqslant |F|$. Hence it follows that

$$|C| \leqslant n + (n-3k+2)\,|F|, \quad \text{because } n \geqslant k\,|F^-| \geqslant 2k. \tag{2}$$

If $n \leqslant 3k-2$, then (1) gives the esssntial upper bound. If $n > 3k-2$, then we may continue (2) using $k\,|F| \leqslant n$:

$$|C| \leqslant n + (n-3k+2)\frac{n}{k} = \frac{n^2}{k} - \left(2 - \frac{2}{k}\right)n. \tag{3}$$

An easy calculation shows that the bound in (1) exceeds that in (3) iff $n < n^*$. This establishes the bounds on L. $\square$

In [10] Liestman and Richards constructed a minimally complete IF with $2(n-k)+1$ calls for the k-uniform hyperstar $S_n^k := (V, \{\{1, \dots, k-1, i\}: i = k, k+1, \dots, n\})$, where minimality follows from the fact that every complete IF on S_n^k has at least $2(n-k)+1$ calls. Consequently, the first bound in Theorem 3.1 turns out to be sharp. To investigate the second case we generalize an idea of Burosch et al. [4] who proved that $L \geq \lceil n^2/4 \rceil$ for $k=2$. Unfortunately there remains a gap of a factor k between the resulting lower and upper bounds for L.

Theorem 3.2. *For $X = Y = V$ and $n \geq 2k$, $L \geq \lfloor n/k \rfloor^2 + 2(n - k\lfloor n/k \rfloor)$.*

Proof. For $k=2$ and n even let φ_n denote the minimally complete IF with $n^2/4$ calls given in [4]. Let be $n = qk + r$ where $0 \leq r \leq k-1$. Split V in $2q$ pairwise disjoint subsets with $\lfloor k/2 \rfloor$ or $\lceil k/2 \rceil$ elements by

$$V_{2i-1} := \left\{(i-1)k+1, \dots, (i-1)k + \left\lfloor \frac{k}{2} \right\rfloor\right\},$$

$$V_{2i} := \left\{(i-1)k + \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, ik\right\}$$

for $i = 1, 2, \dots, q$. We then define an IF ψ by its three phases:

(i) If $r \neq 0$, then carry out the calls $a_x := (\{1, \dots, k-1, qk+x\}: x)$ for $x = 1, 2, \dots, r$.

(ii) The calls of the kernel are organized as φ_{2q} but with the above-defined subsets instead of single points, i.e. the call $(V_{2i-1} \cup V_{2j}; r+t)$ takes place in ψ iff the call $(\{2i-1, 2j\}; t)$ takes place in φ_{2q}. Because $|V_{2i-1} \cup V_{2j}| = k$ this is possible. Let s denote the last step number of the kernel.

(iii) If $r \neq 0$, then carry out the calls $b_x := (\{1, \dots, k-1, qk+x\}; s+x)$ for $x = 1, 2, \dots, r$.

Obviously, ψ is complete since φ_{2q} is complete, and ψ is minimal because by deleting a_x, k does not learn the information of x; and by deleting b_x, the inverse situation occurs. Finally, the kernel does not contain any redundant call because φ_{2q} does not. Altogether we have

$$L(H, \psi) = r + \frac{(2q)^2}{4} + r = q^2 + 2r = \left\lfloor \frac{n}{k} \right\rfloor^2 + \left(n - k\left\lfloor \frac{n}{k} \right\rfloor\right). \qquad \square$$

Fig. 5 shows an RMO of ψ in the case $q=4$, $r=2$, where the calls in the kernel are denoted by the indices of the participating subsets of V. Reading these indices as points, the kernel of ψ in Fig. 5 is φ_8. The generalization for other values of n and k, (resp. q and r) is obvious. In the Hasse diagram, the completeness and minimality of φ_{2q} can easily be checked.

We finish the investigation of long IF with a remark on the number of rounds. While our definition of minimality excludes redundant calls, up to now we have not considered a concept avoiding redundant rounds. It is of interest to consider

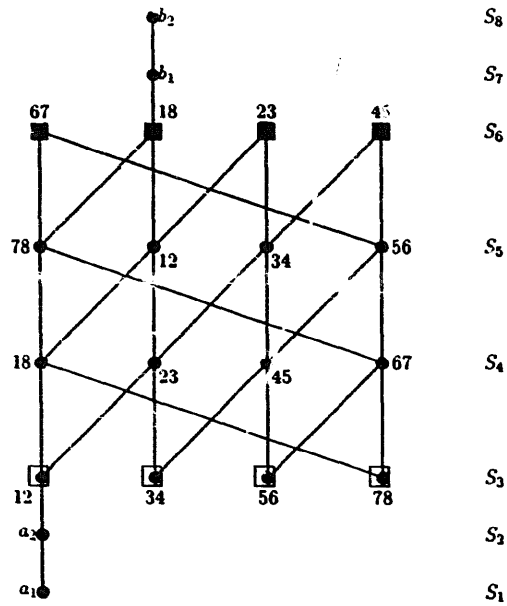

Fig. 5.

IFs which are as "parallelized" as possible. Therefore we propose the following approach: Starting from an (X, Y)-complete IF φ on an hypergraph H consider the MO P_M. Then change the numbers of all calls successively such that:

- S_1 consists of all calls which are minimal elements of P_M,
- a call is carried out immediately after all of its predecessors have been carried out.

Thus a call $d \in C$ gets the number $\operatorname{rank}(d)+1$ where $\operatorname{rank}(d)$ is defined to be the length of a longest ascending chain in P_M ending in d. Let φ^* be this new IF. It is easy to see that there is no IF φ' with a MO isomorphic to that of φ and $T(H, \varphi') < T(H, \varphi^*)$, i.e. φ^* is one of the most effective parallelizations of φ. For the parameter $T = \max_\varphi T((V, \binom{V}{k}), \varphi^*)$ we can conclude the following.

Theorem 3.3.

(a) $\quad T = 2(n-k)+1 \quad$ *if* $n < n^{**}$,

(b) $\quad T \leq n^2/k - 2n + 2 \quad$ *if* $n \geq n^{**}$,

where $n^{**} = 2k + \sqrt{2k^2 - k}$.

Proof. (a) Clearly $T \leq L$. For the hyperstar S_n^k, we have $T(S_n^k, \varphi) = L(S_n^k, \varphi) = 2(n-k)+1$.

(b) The calls of F^- can be done in one round, and so can those of F. Therefore $T(H, \varphi) \leq |\underline{F}^- \setminus F^-| + 1 + |K_0| + 1 + |\bar{F}/F|$, and the assertion follows from the proof of Theorem 3.1. $\square$

It is likely that the bound for the general case (b) in Theorem 3.3 is not realistic and can be improved.

4. Minimum size of kernel-like posets

The remaining part of the paper considers the minimum number of calls that a minimally (X, Y)-complete IF must have. Clearly, the restriction to minimally (X, Y)-complete IFs does not influence this minimal number, but it allows us to start from the structural partitioning of an IF given by the RMO in Section 2. As in Section 3 we consider the three phases of such an IF separately. Because the kernel is the most complicated part, we give a lower bound on its size in this section. In fact, we consider slightly more general posets here, and use the main result for kernels in the next section. We formulate the results and proofs in the language of poset theory and use a notation that is independent of the other sections.

Let $k, l \geqslant 2$ be arbitrarily fixed integers and $P=(M, \leqslant)$ a partial order $\leqslant$ on the set M. Moreover, let $\min M$ and $\max M$ denote the sets of all minimal and maximal elements of P, respectively. The covering relation of P is $\dot{<}$, and for any $x \in M$, define $\operatorname{suc} x$ and $\operatorname{pre} x$ to be the sets of all immediate successors and predecessors of x, respectively. Motivated by the properties of kernels of IFs, proved in Section 2, we consider posets P satisfying the following conditions:

$$\text{For any } x \in M,\ |\operatorname{suc} x| \leqslant l \text{ and } |\operatorname{pre} x| \leqslant k. \tag{I}$$

$$\text{For any } u \in \min M,\ v \in \max M,\ u \leqslant v. \tag{II}$$

Let the cardinalities $p := |\min M|$ and $q := |\max M|$ be fixed. Then the question arises as to how small such a poset can be.

Obviously, the inverse order of P fulfills conditions (I) and (II) iff P, with exchanged parameters k and l, does. Hence we may assume that

$$\frac{p-k}{k-1} \geqslant \frac{q-l}{l-1}$$

w.l.o.g. Moreover we will restrict ourselves to the nontrivial case $p, q, \geqslant 2$. The main result is the following.

Theorem 4.1. *For any poset P which satisfies* (I) *and* (II) *and has exactly p minimal and q maximal elements,*

$$|M| \geqslant m := p + \left\lceil \frac{p-k}{k-1} \right\rceil + q + \left\lceil \frac{q-l}{l-1} \right\rceil.$$

The proof is divided in several parts. Suppose that $|M| < m$.

Proposition 4.1. $\min M \cap \max M = \emptyset$.

Proof. Let $u \in \min M \cap \max M$. Since $p \geq 2$, we have an element $u' \in \min M$, $u' \neq u$, with $u' \leq u \in \max M$ by (II). This contradicts the fact that $u \in \min M$. □

Our next aim is to extend the covering relation of P to a relation R with analogous properties but in which each nonmaximal element has exactly k predecessors. To do this, add a set $\min 'M$ of kp new elements to M. The covering relation of P is extended by the definition that in R every element of $\min M$ succeeds exactly k distinct elements of $\min 'M$, but every element of $\min 'M$ precedes exactly one element of $\min M$. Furthermore, for any $x \in M_0 := M \setminus (\min M \cup \max M)$, define elements of $\min 'M$ to be predecessors of x in R such that in R each $x \in M_0$ has exactly k predecessors and each element of $\min 'M$ has at most one successor in M_0. To do this, the kp elements of $\min 'M$ suffice, since

$$|M_0| = |M| - |\min M| - |\max M| < m - p - q \leq 2\left\lceil \frac{p-k}{k-1} \right\rceil .$$

Each $x \in M_0$ needs at most $k-1$ additional predecessors in $\min 'M$, and

$$|M_0|\,(k-1) < 2\left(\frac{p-k}{k-1} + 1\right)(k-1) = 2p - 2 < kp = |\min 'M|.$$

Throughout the rest of the proof we consider the transitive closure P' of this relation R on the set $M' := M \cup \min 'M$. For the sake of simplicity we will also use the symbol $x \leq y$ iff $(x, y) \in P'$ for elements $x, y \in M'$. Note that eventually the covering relation of P' differs from R. Therefore, if $(x, y) \in R$, then we will say that x is an R-predecessor of y, and y is an R-successor of x in P'. The following properties are immediate consequences of the above construction:

(4.1) The q elements of $\max M$ are the maximal elements of P'.

(4.2) The kp elements of $\min 'M$ are the minimal elements of P'.

(4.3) Each element has no more than l R-successors in P'.

(4.4) Each element of $\max M$ has at most k, each element of $M_0' := M_0 \cup \min M$ has exactly k R-predecessors in P'.

(4.5) For any $z \in \min 'M$ and $v \in \max M$, $z < v$ because there is a $u \in \min M$ with $z < u$ by construction and $u \leq v$ by (II).

(4.6) $$|M'| = |M| + |\min 'M| < m + kp = kp + \left\lceil \frac{p-k}{k-1} + p \right\rceil + q + \left\lceil \frac{q-l}{l-1} \right\rceil$$
$$= kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + q + \left\lceil \frac{q-l}{l-1} \right\rceil .$$

The rest of the proof generalizes that of Kleitman and Shearer in [7] where they establish a lower bound for complete IFs on k-uniform hypergraphs.

For any $x \in M'$, let $\underline{x}$ or $\bar{x}$ denote the lower or upper ideal generated by x in P', respectively.

Proposition 4.2. *For any $x \in M'$, if* $\min{}'M \subseteq \underline{x}$ *or* $\max M \subseteq \bar{x}$, *then*

$$|\underline{x}| \geqslant kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + 1$$

or

$$|\bar{x}| \geqslant q + \left\lceil \frac{q-l}{l-1} \right\rceil + 1,$$

respectively.

Proof. Both parts can be proved in the same way. Let us consider the first inequality. Note that $x \notin \min{}'M$ because otherwise $|\underline{x}| = |\{x\}| = 1 < 4 \leqslant kp = |\min{}'M|$, which would contradict the assumption that $\min{}'M \subseteq \underline{x}$. Let $N := \underline{x} \setminus (\min{}'M \cup \{x\})$. For the set Q of all pairs $(y, z) \in R$ with $y, z \in \underline{x}$ we have $|Q| \leqslant (|N| + 1)k$, because each nonminimal element has at most k R-predecessors. Moreover, $|Q| \geqslant kp + |N|$ because each element of $\underline{x} \setminus \{x\}$ has at least one R-successor in $\underline{x}$. Hence

$$|\underline{x}| = kp + |N| + 1 \geqslant kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + 1. \qquad \square$$

Now let $\mathcal{I}$ be the set of all lower ideals in P' with exactly

$$kp + \left\lceil \frac{kp-k}{k-1} \right\rceil$$

elements.

Proposition 4.3. *For any $I \in \mathcal{I}$,*

(a) $I \cap \max M = \emptyset$,

(b) $\min{}'M \subseteq I$.

Proof. (a) Assume that there is $v \in I \cap \max M$. Then also $\underline{v} \subseteq I$, but by (4.5) we have $\min{}'M \subseteq \underline{v}$ and by Proposition 4.2,

$$|\underline{v}| \geqslant kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + 1 > |I|.$$

(b) Assume that there is $u \in \min{}'M$, $u \notin I$. Then also $\bar{u} \cap I = \emptyset$ and consequently, $|M'| \geqslant |I| + |\bar{u}|$. By (4.5) we know $\max M \subseteq \bar{u}$, and by Proposition 4.2,

$$|\bar{u}| \geqslant q + \left\lceil \frac{q-l}{l-1} \right\rceil + 1.$$

Hence,

$$|M'| \geqslant kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + q + \left\lceil \frac{q-l}{l-1} \right\rceil + 1,$$

which contradicts (4.6). $\square$

For each subset $N \subseteq M'$ let $G(N)$ denote the Hasse diagram of the restriction of R to N. If a component of $G(N)$ turns out to be a tree, then we will call it a tree-component. Moreover, let $|G|$ denote the cardinality of nodes in the graph G.

Proposition 4.4. *For any $I \in \mathcal{I}$, $G(I)$ contains at least two tree-components.*

Proof. We show that $G(I)$ has not more than $|G(I)|-2$ edges. Because of Proposition 4.3, I consists of $\min{}'M$ and $|I|-kp$ nonmaximal elements of M' each having exactly k R-predecessors. Hence $G(I)$ has

$$k\left\lceil \frac{kp-k}{k-1} \right\rceil < kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + (k-1)\left(\frac{kp-k}{k-1}+1\right) - kp = |G(I)| - 1$$

edges. $\square$

Among all tree-components of $G(I)$ we choose the two smallest with respect to the number of points and denote them by $T_1(I)$ and $T_2(I)$, where $|T_1(I)| \leqslant |T_2(I)|$. Then among all $I \in \mathcal{I}$, we can choose an I' such that $T_1(I') \cup T_2(I')$ has minimum size. Finally, among all $I \in \mathcal{I}$ with

$$|T_1(I)| + |T_2(I)| = |T_1(I')| + |T_2(I')|$$

we select an $J \in \mathcal{I}$ such that $T_1(J)$ has minimum size. For $i = 1, 2$, let z_i be an arbitrarily fixed maximal element among the elements of $T_i(J)$.

Proposition 4.5. $\max M \subseteq \bar{z}_1$.

Proof. If $z_1 \in \min{}'M$, then the assertion follows from (4.5). Let us assume $z_1 \notin \min{}'M$ in the following. Then also $z_2 \notin \min{}'M$ since $|T_1(J)| \leqslant |T_2(J)|$.

For $i = 1, 2$, by Proposition 4.3 we also have $z_i \notin \max M$, i.e. $z_i \in M_0 \cup \min M$. By (4.4), z_i has exactly k R-predecessors. Hence, after deleting z_i with all the incident edges, the tree $T_i(J)$ splits into k subtrees, which we denote by $T_{ij}(J)$, $j = 1, 2, \ldots, k$.

The idea is to prove that every minimal element of $M' \backslash J$ is $\geqslant z_1$. Then $M' \backslash J \subseteq \bar{z}_1$, and we are done because $\max M \subseteq M' \backslash J$ by Proposition 4.3.

Assume that we have an element y which is minimal in $M' \backslash J$ and for which $z_1 \not\leqslant y$. Then $J' := (J \backslash \{z_1\}) \cup \{y\}$ is another lower ideal of $\mathcal{I}$. Consider the tree-components of $G(J')$. We start from the fact that at least $T_{11}(J), \ldots, T_{1k}(J)$

and $T_2(J)$ are tree-components of $G(J\setminus\{z_1\})$. Now consider the different situations that occur by putting in the element y.

Case 1: *Two of the $T_{1j}(J)$ remain tree-components of $G(J')$, w.l.o.g. $T_{11}(J)$ and $T_{12}(J)$.*

Then

$$|T_1(J')| + |T_2(J')| \leq |T_{11}(J)| + |T_{12}(J)| < |T_1(J)| < |T_1(J)| + |T_2(J)|,$$

which contradicts the minimality of $T_1(J) \cup T_2(J)$.

Case 2: *Exactly one of the $T_{1j}(J)$ remains a tree-component of $G(J')$, w.l.o.g. $T_{11}(J)$.*

If also $T_2(J)$ is a tree-component of $G(J')$, then

$$|T_1(J')| + |T_2(J')| \leq |T_{11}(J)| + |T_2(J)| < |T_1(J)| + |T_2(J)|,$$

which again contradicts our assumption. If $T_2(J)$ is not a tree-component of $G(J')$, then the k edges of y in $G(J')$ must join $T_2(J)$ and $T_{12}(J), \ldots, T_{1k}(J)$ to a tree-component of $G(J')$. Thus

$$\begin{aligned}|T_1(J')| + |T_2(J)')| &\leq |T_{11}(J) \cup (T_2(J) \cup T_{12}(J) \cup \cdots \cup T_{1k}(J) \cup \{y\})| \\ &= |T_1(J)| + |T_2(J)|.\end{aligned}$$

Because of the minimality of $|T_1(J)| + |T_2(J)|$, equality holds above, and finally $|T_1(J')| \leq |T_{11}(J)| < |T_1(J)|$, which contradicts the minimality of $T_1(J)$.

Case 3: *None of the $T_{1j}(J)$ is a tree-component of $G(J')$.*

Then the k edges of y in $G(J')$ have to join $T_{11}(J), \ldots, T_{1k}(J)$ and therefore z_2 and y are incomparable in P'. Hence $J'' := (J\setminus\{z_2\}) \cup \{y\} \in \mathcal{J}$ and each $T_{2j}(J)$ is a tree-component of $G(J'')$. Therefore

$$|T_1(J'')| + |T_2(J'')| < |T_{21}(J)| + |T_{22}(J)| < |T_2(J)| < |T_1(J)| + |T_2(J)|,$$

which contradicts the minimality of $T_1(J) \cup T_2(J)$. $\square$

Consequently, $z_1 \leq y$ for any minimal element of $M'\setminus J$. Since z_1 is maximal among all elements of J, we have $J \cap \bar{z}_1 = \{z_1\}$, and therefore $|M'| \geq |J \cup \bar{z}_1| = |J| + |\bar{z}_1| - 1$. Because of Proposition 4.5 we may apply Proposition 4.2 to $\bar{z}_1$. This yields

$$|M'| \geq kp + \left\lceil \frac{kp-k}{k-1} \right\rceil + q + \left\lceil \frac{q-l}{l-1} \right\rceil + 1 - 1,$$

which contradicts (4.6). Hence, our first assumption that $|M| < m$ is false, and Theorem 4.1 is proved.

Although we do not need it later we should mentioned here that this bound cannot be improved.

Theorem 4.2. *For $p \geq q \geq 2$, there is a poset P of*

$$p + \left\lceil \frac{p-k}{k-1} \right\rceil + q + \left\lceil \frac{q-l}{l-1} \right\rceil$$

elements which has exactly p minimal and q maximal elements and satisfies (I) *and* (II).

Proof. On the set $\{1, 2, \ldots, p + \lceil (p-k)/(k-1) \rceil\}$ define a relation Q by:
if $p \leq k$, then $Q = \emptyset$;
if $k < p \leq 2k-1$, then $Q := \{(i, p+1): i = 1, \ldots, k\}$;
if $p > 2k-1$, then

$$Q := \{(i, p+1): i = 1, \ldots, k\} \cup \{(p+j-1, p+j), (i, p+j):$$
$$j = 2, \ldots, \left\lceil \frac{p-k}{k-1} \right\rceil; i = (j-1)(k-1)+2, \ldots, j(k-1)+1\}$$

Note that

$$p_0 := \left\lceil \frac{p-k}{k-1} \right\rceil (k-1) + 1 < \left(\frac{p-k}{k-1} + 1 \right)(k+1) + 1 = p.$$

It is easy to see that each element has no more than one successor and no more than k predecessors. Furthermore, there are p minimal elements $1, \ldots, p$ and no more than k maximal elements, i.e. if $p \leq k$, then $1, \ldots, p$, or otherwise, $p_0 + 1, \ldots, p, p + \lceil (p-k)/(k-1) \rceil$.

Using the inverse relation, with appropriately modified parameters, we get a relation Q' with exactly q maximal and no more than l minimal elements. This can be joined to Q by defining each minimal element of Q' to cover each maximal element of Q. The transitive closure is a poset P of the asserted kind. □

For the sake of completeness we should mention the results for the trivial cases which have been excluded up to now. For $p = q = 1$, the minimal value of $|M|$ obviously equals 1. If $p > q = 1$, then we have

$$|M| \geq p + \left\lceil \frac{p-k}{k-1} \right\rceil + 1$$

as in Proposition 4.2. By adding one new maximal element to the above-constructed relation Q we get a poset achieving this bound.

5. Short information flows

In this section we present a lower bound for the smallest possible number l of calls a minimally (X, Y)-complete IF φ must have. Let φ be given arbitrarily on

$H=(V, \binom{V}{k})$. We use the notation introduced in Section 2 for the RMO of φ, and follow the same approach as in Section 3.

Lemma 5.1.

$$|\underline{F}^-\backslash F^-| \geqslant \left\lceil \frac{|X|-k\,|F^-|}{k-1} \right\rceil, \qquad |\bar{F}\backslash F| \geqslant \left\lceil \frac{|Y|-k\,|F|}{k-1} \right\rceil.$$

Proof. Each of the $k\,|F|$ participants of an irredundant F-call already knows the complete information after F. Therefore, at least $|Y|-k\,|F|$ points must learn it in $\bar{F}\backslash F$. On the other hand, for any call $d \in \bar{F}\backslash F$, there is at least one participant of d who knew everything before d because $d \cap p(d) \neq \emptyset$. Hence $(k-1)\,|\bar{F}\backslash F| \geqslant |Y|-k\,|F|$. The proof for $\underline{F}^-\backslash F^-$ is similar. $\square$

Theorem 5.1. *For non-empty $X, Y \subseteq V$,*

$$l \geqslant \left\lceil \frac{|X|-k}{k-1} \right\rceil + \left\lceil \frac{|Y|-k}{k-1} \right\rceil.$$

Proof. For any minimally (X, Y)-complete IF φ on H we have

$$L(H, \varphi) = |C| = |\underline{F}^-\backslash F^-| + |K| + |\bar{F}\backslash F|.$$

If $|F^-|=1$, then also $|F|=|K|=1$ by Theorem 2.1, and from Lemma 5.1 we have:

$$|C| \geqslant \left\lceil \frac{|X|-k}{k-1} \right\rceil + \left\lceil \frac{|Y|-k}{k-1} \right\rceil + 1.$$

If $|F^-| \geqslant 2$, then the kernel is a poset of the type discussed in Section 4. It has properties (I) and (II), and the sets F^- and F are the sets of all minimal and maximal elements, respectively. Therefore, by Theorem 4.1:

$$|K| \geqslant |F^-| + \left\lceil \frac{|F^-|-k}{k-1} \right\rceil + |F| + \left\lceil \frac{|F|-k}{k-1} \right\rceil.$$

Using Lemma 5.1 again it follows that:

$$\begin{aligned}
|C| &\geqslant \left\lceil \frac{|X|-k\,|F^-|}{k-1} \right\rceil + |F^-| + \left\lceil \frac{|F^-|-k}{k-1} \right\rceil + \left\lceil \frac{|F|-k}{k-1} \right\rceil + |F| + \left\lceil \frac{|Y|-k\,|F|}{k-1} \right\rceil \\
&\geqslant \left\lceil \frac{1}{k-1}(|X|-k\,|F^-|-(k-1)\,|F^-|+|F^-|-k) \right\rceil \\
&\quad + \left\lceil \frac{1}{k-1}(|F|-k+(k-1)\,|F|+|Y|-k\,|F|) \right\rceil \\
&= \left\lceil \frac{|X|-k}{k-1} \right\rceil + \left\lceil \frac{|Y|-k}{k-1} \right\rceil. \qquad \square
\end{aligned}$$

The exact value of l depends on the mutual position of X and Y in V. If they have a large intersection, then a well-known construction ([7, 10]) can be generalized to our situation.

Theorem 5.2. *For $X, Y \subseteq V$ with $|X \cap Y| \geqslant k^2$,*

$$l = \left\lceil \frac{|X| - k}{k-1} \right\rceil + \left\lceil \frac{|Y| - k}{k-1} \right\rceil.$$

Proof. Select a set $Z = \{x_{ij}: i, j = 1, 2, \ldots, k\}$ of k^2 points of $X \cap Y$. Then the IF consists of:

(1) redundant F^--calls: $X \setminus Z$ is divided in $\lceil (|X| - k^2)/(k-1) \rceil$ groups of no more than $k-1$ elements. To each of these groups we joint points of Z such that each group contains exactly k elements which carry out a call. After this, the k^2 points of Z have collected all of the information.

(2) The irredundant F^--calls take place on the edges $\{x_{ij}: j = 1, \ldots, k\}$, for $i = 1, 2, \ldots, k$.

(3) The irredundant F-calls take place on the edges $\{x_{ij}: i = 1, \ldots, k\}$, for $j = 1, 2, \ldots, k$. Now each point of Z knows all of the information. Note that the inner kernel is empty.

(4) The redundant F-calls are called out just as the calls of the first step, but with respect to Y instead of X. Obviously then the IF is (X, Y)-complete.

Altogether this IF has

$$\left\lceil \frac{|X| - k^2}{k-1} \right\rceil + k + k + \left\lceil \frac{|Y| - k^2}{k-1} \right\rceil = \left\lceil \frac{|X| - k}{k-1} \right\rceil + \left\lceil \frac{|Y| - k}{k-1} \right\rceil$$

calls. $\square$

For other special cases, these inequalities yield additional lower bounds.

Theorem 5.3. *For non-empty $X, Y \subseteq V$ with $|X| \leqslant k+1$ or $|Y| \leqslant k+1$,*

$$l \geqslant \left\lceil \frac{|X| - k}{k-1} \right\rceil + \left\lceil \frac{|Y| - k}{k-1} \right\rceil + 1.$$

Proof. Modify the proof of Theorem 5.1 in the case $|F^-|, |F| \geqslant 2$. For example, if $|Y| \leqslant k+1$, then

$$\left\lceil \frac{|Y| - k\,|F|}{k-1} \right\rceil \leqslant \left\lceil \frac{k+1-2k}{k-1} \right\rceil \leqslant -1,$$

i.e.,

$$|\bar{F} \setminus F| \geqslant 0 \geqslant \left\lceil \frac{|Y| - k\,|F|}{k-1} \right\rceil + 1.$$

Using this instead of Lemma 5.1 gives the asserted result. $\square$

The important 'classical' case $X = Y = V$ was solved completely before ([1, 7, 9]). While the result for $n \geqslant k^2$ is contained in our Theorem 5.2, the well-known lower bound

$$l \geqslant \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil$$

can also be computed here following the proof of Theorem 5.1: If $|F^-| = 1$, then

$$|C| \geqslant 2\left\lceil \frac{n-k}{k-1} \right\rceil + 1 = \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n-1}{k-1} \right\rceil \geqslant \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil.$$

If $|F^-| \geqslant 2$, then

$$\begin{aligned} |C| &\geqslant \left\lceil \frac{n-k\,|F^-|}{k-1} \right\rceil + |F^-| + |F| + \left\lceil \frac{n+k\,|F|}{k-1} \right\rceil \\ &= \left\lceil \frac{n-|F^-|}{k-1} \right\rceil + \left\lceil \frac{n-|F|}{k-1} \right\rceil \geqslant \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil \end{aligned}$$

because $|F^-| \leqslant k$ and $k\,|F| \leqslant n$. For $k = 2$, the upper bound $l \leqslant |X| + |Y| - 4$ was shown in [11].

6. Existence of kernel-like posets

In this last section we continue the investigation in Section 4 and use the notation introduced there. It is motivated by the fact that the poset constructed in the proof of Theorem 4.2 cannot be the kernel of an IF if $p > 2k - 1$ because Proposition 2.5(i) does not hold. So e.g. every maximal element of Q is smaller than any maximal element of the whole poset P. Therefore we should introduce a new condition.

$$\begin{aligned} &\text{For any } x \in M\setminus(\min M \cup \max M)\text{, there are elements } u \in \min M \\ &\text{and } v \in \max M \text{ such that } u \nleqslant x \text{ and } v \ngeqslant x. \end{aligned} \tag{III}$$

Moreover, by Proposition 2.7 we may restrict condition (I) to the case $k = l$ throughout this section.

It is easy to see that, if $p, q \leqslant k$, then the construction in Theorem 4.2 yields a poset satisfying (I), (II), (III) with as few elements as possible, by Theorem 4.1. Unfortunately, we cannot answer the similar question in general. A partial answer is given in the following result.

Theorem 6.1. *If $p = j(k-1) + 2$ for some positive integer j, then there is a poset P on*

$$2\left(p + \left\lceil \frac{p-k}{k-1} \right\rceil\right)$$

elements which has exactly p minimal and p maximal elements and fulfills (I), (II), (III).

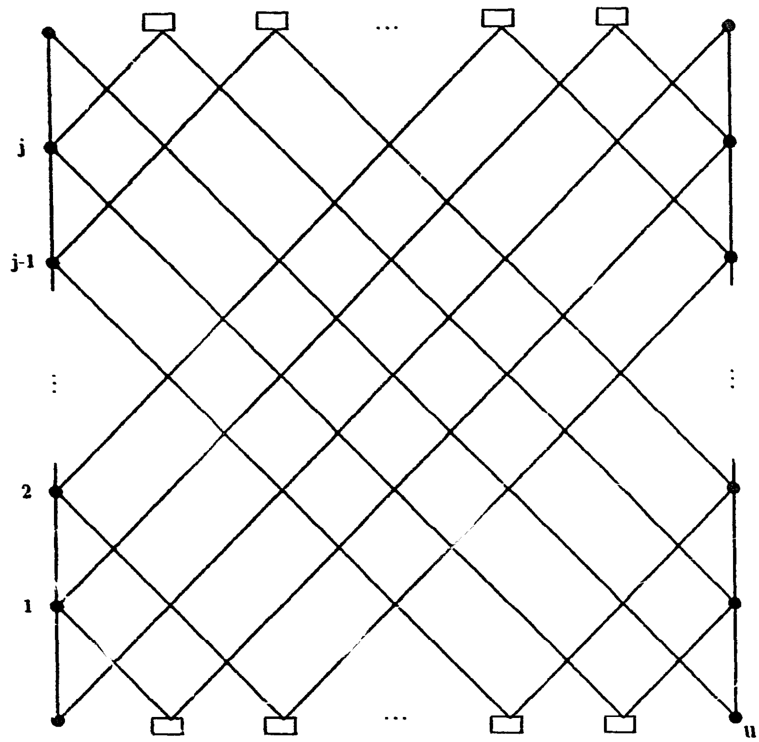

Fig. 6.

Proof. We avoid the formal definition and give the Hasse diagram in Fig. 6. Here • indicates a single element, □, an antichain of $k-1$ elements, $\overset{|}{\square}$ indicates each of the $k-1$ elements of □ is covered by •, and $\underset{|}{\square}$ indicates the inverse situation.

(I) and (II) are easy to see. Because none of the elements $1, 2, \ldots, j$ is greater than u and because of symmetry, (III) also holds. Finally, F has $p = j(k-1)+2$ minimal and maximal elements and altogether it consists of $2(p+j)$ elements. By $0 < 1/(k-1) < 1$ and

$$\frac{p-k}{k-1} = \frac{p-2}{k-1} - 1 + \frac{1}{k-1} = j - 1 + \frac{1}{k-1},$$

we have

$$\left\lceil \frac{p-k}{k-1} \right\rceil = j. \qquad \square$$

Similar constructions exist, if $p = 2k-1$ or $p = 2k+m$, $1 \leqslant m \leqslant k-2$. If $k=2$, then Theorem 6.1 holds for all p and the constructed posets turn out to be exactly the kernels of the NOHO-IF introduced and investigated by West [13]. Thus our construction generalizes this concept to $k > 2$, but unfortunately not for all p. The next theorem shows that a poset of the considered kind cannot exist for all parameters p and q.

Theorem 6.2. *If* (a) $q \leqslant k < p$ *or* (b) $p \geqslant kq$, *then there is no poset on*

$$p + \left\lceil \frac{p-k}{k-1} \right\rceil + q + \left\lceil \frac{q-k}{k-1} \right\rceil$$

elements which has exactly p minimal and q maximal elements and satisfies (I), (II), (III).

Proof. Assume that we have a poset P of this type.

(a) It suffices to prove that $q \leqslant k$ implies $p \leqslant k$. For $v \in \max M$, we have $\min M \subseteq \underline{v}$ because of (II), and similar to Proposition 4.2,

$$|\underline{v}| = p + \left\lceil \frac{p-k}{k-1} \right\rceil + 1,$$

i.e.,

$$|\underline{v} \setminus \{v\}| = p + \left\lceil \frac{p-k}{k-1} \right\rceil.$$

On the other hand,

$$|M \setminus \max M| = p + \left\lceil \frac{p-k}{k-1} \right\rceil,$$

since $|\max M| = q$, $\lceil (q-k)/(k-1) \rceil = 0$. But $\underline{v} \setminus \{v\} \subseteq M \setminus \max M$, since $\max M \cap \underline{v} = \{v\}$. Consequently, $\underline{v} \setminus \{v\} = M \setminus \max M$. For fixed $x \in M \setminus \max M$, this means $x \leqslant v$ for any $v \in \max M$. By (III) we get $x \in \min M$, i.e. $\min M = M \setminus \max M$ because of Proposition 4.1. Therefore,

$$p = |\min M| = |M \setminus \max M| = p + \left\lceil \frac{p-k}{k-1} \right\rceil,$$

and this implies $\lceil (p-k)/(k-1) \rceil = 0$ and $p \leqslant k$.

(b) As in the proof of Theorem 4.1 we consider the Hasse diagram of P and determine the number r of edges in the subgraph on $M \setminus \min M$. Because each element of $M \setminus \min M$ has at most k predecessors, $G(M)$ contains no more than $k\,|M \setminus \min M|$ edges. If any element of $\min M$ has exactly one successor x, then x is smaller than each element of $\max M$, by (II). But this contradicts (III), and hence every element of $\min M$ has at least 2 successors. Therefore, at least $2p$ edges of $G(M)$ are not in $G(M \setminus \min M)$, i.e.

$$r \leqslant k\left(\left\lceil \frac{p-k}{k-1} \right\rceil + q + \left\lceil \frac{q-k}{k-1} \right\rceil\right) - 2p.$$

In (a) we used the fact that for any $v \in \max M$, $|\underline{v}| \geqslant p + \lceil (p-k)/(k-1) \rceil + 1$. Hence, for $M_0 = M \setminus (\min M \cup \max M)$, we have

$$|M_0 \cap \underline{v}| \geqslant \left\lceil \frac{p-k}{k-1} \right\rceil,$$

and consequently every component of $G(M\setminus\min M)$ must contain no less than $\lceil(p-k)/(k-1)\rceil$ elements of M_0. By

$$|M_0| = \left\lceil\frac{p-k}{k-1}\right\rceil + \left\lceil\frac{q-k}{k-1}\right\rceil$$

there are at most

$$1+\left\lfloor\left\lceil\frac{q-k}{k-1}\right\rceil \Big/ \left\lceil\frac{p-k}{k-1}\right\rceil\right\rfloor \leqslant 2$$

components. Because $G(M\setminus\min M)$ consists of

$$\left\lceil\frac{p-k}{k-1}\right\rceil + q + \left\lceil\frac{q-k}{k-1}\right\rceil$$

points, it follows that

$$r \geqslant \left\lceil\frac{p-k}{k-1}\right\rceil + q + \left\lceil\frac{q-k}{k-1}\right\rceil - 2.$$

This finally yields

$$\begin{aligned} 2p-2 &\leqslant (k-1)\left(\left\lceil\frac{p-k}{k-1}\right\rceil + q + \left\lceil\frac{q-k}{k-1}\right\rceil\right) \\ &< (k-1)\left(\frac{p-k}{k-1} + 1 + q + \frac{q-k}{k-1} + 1\right) = p + kq - 2 \end{aligned}$$

and $p < kq$. Hence for $p \geqslant kq$ a poset of the required kind cannot exist. □

Theorem 6.2 shows that the numbers of minimal and maximal elements cannot differ too much. In fact, we conjecture that such posets exist only if $p = q$. In the original telephone problem, this last result can be interpreted as follows. In any complete IF with as few calls as possible, the number of irredundant F^- and F-calls are not too different, and we conjecture that they are equal.

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